

# Hunt's Hypothesis (H) for the Sum of Two Independent Lévy Processes

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**Abstract** Which Lévy processes satisfy Hunt's hypothesis (H) is a long-standing open problem in probabilistic potential theory. The study of this problem for one-dimensional Lévy processes suggests us to consider (H) from the point of view of the sum of Lévy processes. In this paper, we present theorems and examples on the validity of (H) for the sum of two independent Lévy processes. We also give a novel condition on the Lévy measure which implies (H) for a large class of one-dimensional Lévy processes.

**Keywords** Hunt's hypothesis (H), Gettoors conjecture, Lévy process.

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# 1 Introduction

Let  $X$  be a time-homogeneous Markov process. Hunt's hypothesis (H) says that “every semipolar set of  $X$  is polar”. This hypothesis plays a crucial role in probabilistic potential theory. In particular, it is equivalent to many important principles of potential theory under mild conditions. These include the bounded positivity principle, bounded energy principle, bounded maximum principle and the bounded regularity principle (see e.g. [10, Proposition 1.1]).

In spite of its importance, (H) has been verified only in special situations. About fifty years ago, Professor R.K. Gettoor conjectured that essentially all Lévy processes satisfy (H). This conjecture stills remains open and is a major unsolved problem in the potential theory for Lévy processes (cf. [1, page 70]).

In the following, we will use a diagram to summarize some sufficient conditions that obtained so far for the validity of (H) for Lévy processes. Let  $(\Omega, \mathcal{F}, P)$  be a probability space and  $X = (X_t)_{t \geq 0}$  be an  $\mathbf{R}^n$ -valued Lévy process on  $(\Omega, \mathcal{F}, P)$  with Lévy-Khintchine exponent  $\psi$ , i.e.,

$$E[\exp\{i\langle z, X_t \rangle\}] = \exp\{-t\psi(z)\}, \quad z \in \mathbf{R}^n, \quad t \geq 0.$$

Hereafter  $E$  denotes the expectation w.r.t. (with respect to)  $P$ ,  $\langle \cdot, \cdot \rangle$  and  $|\cdot|$  denote respectively the Euclidean inner product and norm of  $\mathbf{R}^n$ . The classical Lévy-Khintchine formula tells us that

$$\psi(z) = i\langle a, z \rangle + \frac{1}{2}\langle z, Qz \rangle + \int_{\mathbf{R}^n} (1 - e^{i\langle z, x \rangle} + i\langle z, x \rangle 1_{\{|x| < 1\}}) \mu(dx),$$

where  $a \in \mathbf{R}^n$ ,  $Q$  is a symmetric nonnegative definite  $n \times n$  matrix, and  $\mu$  is a measure (called the Lévy measure) on  $\mathbf{R}^n \setminus \{0\}$  satisfying  $\int_{\mathbf{R}^n \setminus \{0\}} (1 \wedge |x|^2) \mu(dx) < \infty$ .

We use  $\operatorname{Re}(\psi)$  and  $\operatorname{Im}(\psi)$  to denote respectively the real and imaginary parts of  $\psi$ , and use also  $(a, Q, \mu)$  to denote  $\psi$ . Define

$$A := 1 + \operatorname{Re}(\psi), \quad B := |1 + \psi|.$$

For a finite (positive) measure  $\nu$  on  $\mathbf{R}^n$ , we denote

$$\hat{\nu}(z) := \int_{\mathbf{R}^n} e^{i\langle z, x \rangle} \nu(dx).$$

$\nu$  is said to have finite 1-energy if

$$\int_{\mathbf{R}^n} \frac{A(z)}{B^2(z)} |\hat{\nu}(z)|^2 dz < \infty.$$

Throughout this paper, we use  $\log$  to denote  $\log_e$ .

We state below the various sufficient conditions for the validity of (H) for Lévy processes.

(ND):  $Q$  is non-degenerate, i.e., the rank of  $Q$  equals  $n$ .

(KF):  $X$  has resolvent densities w.r.t. the Lebesgue measure and the Kanda-Forst condition holds, i.e.,  $|\operatorname{Im}(\psi)| \leq cA$  for some constant  $c > 0$ .

(R):  $X$  has resolvent densities w.r.t. the Lebesgue measure and Rao's condition holds, i.e.,  $|\operatorname{Im}(\psi)| \leq Af(A)$ , where  $f$  is a positive increasing function on  $[1, \infty)$  such that  $\int_N^\infty (\lambda f(\lambda))^{-1} d\lambda = \infty$  for some  $N \geq 1$ .

(EKFR):  $X$  has resolvent densities w.r.t. the Lebesgue measure and the following extended Kanda-Forst-Rao condition holds:

There are two measurable functions  $\phi_1$  and  $\phi_2$  on  $\mathbf{R}^n$  such that  $\operatorname{Im}\psi = \phi_1 + \phi_2$ , and

$$|\phi_1| \leq Af(A), \quad \int_{\mathbf{R}^n} \frac{|\phi_2(z)|}{B^2(z)} dz < \infty,$$

where  $f$  is a positive increasing function on  $[1, \infty)$  such that  $\int_N^\infty (\lambda f(\lambda))^{-1} d\lambda = \infty$  for some  $N \geq 1$ .

( $C^{B/A}$ ):  $X$  has resolvent densities w.r.t. the Lebesgue measure and there exists a constant  $c > 0$  such that  $B(z) \leq cA(z) \log(2 + B(z)) [\log \log(2 + B(z))]$ ,  $\forall z \in \mathbf{R}^n$ .

( $C^0$ ):  $X$  has resolvent densities w.r.t. the Lebesgue measure and for any finite measure  $\nu$  on  $\mathbf{R}^n$  of finite 1-energy,

$$\int_{\mathbf{R}^n} \frac{1}{B(z) \log(2 + B(z)) [\log \log(2 + B(z))]} |\hat{\nu}(z)|^2 dz < \infty.$$

(SYM):  $X$  has resolvent densities w.r.t. the Lebesgue measure and is symmetric.

(SP):  $X$  has bounded continuous transition densities, and  $X$  and its symmetrization have the same polar sets.

(S):  $\mu(\mathbf{R}^n \setminus \sqrt{Q}\mathbf{R}^n) < \infty$  and the following solution condition holds:

The equation  $\sqrt{Q}y = -a - \int_{\mathbf{R}^n \setminus \sqrt{Q}\mathbf{R}^n} x 1_{\{|x|<1\}} \mu(dx)$  has at least one solution  $y \in \mathbf{R}^n$ .

Now we can present the diagram that summarizes all the above sufficient conditions for the validity of (H) for Lévy processes.

$$\begin{array}{ccccccc}
 & & (SYM) & & & & \\
 & & \Downarrow & & & & \\
 (ND) & \Rightarrow & (KF) & \Rightarrow & (C^{B/A}) & \Rightarrow & (C^0) \\
 & & \Downarrow & & & & \Downarrow \\
 & & (R) & \Rightarrow & (EKFR) & \Rightarrow & (H) \Leftarrow (SP) \\
 & & & & & & \Uparrow \\
 & & & & & & (S)
 \end{array}$$

We refer the readers to [11, 5, 13, 8, 10, 9] for the proof of the diagram. We also refer the readers to [6] and [4] for recent interesting results on the validity of (H). In [6], Hansen and Netuka showed that (H) holds if there exists a Green function  $G > 0$  which locally satisfies the triangle inequality  $G(x, z) \wedge G(y, z) \leq CG(x, y)$ . In [4], Fitzsimmons showed that Gross's Brownian motion, which is an infinite-dimensional Lévy process, fails to satisfy (H).

In this paper, we will further study Hunt's hypothesis (H) from the point of view of the sum of two independent Lévy processes. The rest of the paper is organized as follows. In Section 2, we discuss (H) for one-dimensional Lévy processes and provide motivation for exploring (H) through considering sums of Lévy processes. Theorem 2.2 below extends a result of Kesten [12], and Theorem 2.3 below presents a novel condition on the Lévy measure  $\mu$  which implies (H) for a large class of one-dimensional Lévy processes. In Section 3, we consider (H) for the sum of two independent Lévy processes without assuming that resolvent densities exist. We show that if  $X_1$  satisfies (H) and  $X_2$  is a compound Poisson process, then  $X_1 + X_2$  satisfies (H); and that if both  $X_1$  and  $X_2$  satisfy condition (S), then  $X_1 + X_2$  satisfies (H). In Section 4, we consider (H) for the sum of two independent Lévy processes under the assumption that resolvent densities exist. Roughly speaking, the results imply that if  $X_1$  satisfies (H) and  $X_2$  is suitably controlled by  $X_1$ , then  $X_1 + X_2$  satisfies (H).

## 2 (H) for one-dimensional Lévy processes

In this section, we consider Hunt's hypothesis (H) for one-dimensional Lévy processes. Let  $X = (X_t)_{t \geq 0}$  be a Lévy process on  $\mathbf{R}$  with Lévy-Khintchine exponent  $\psi$  and  $(a, Q, \mu)$ , where  $Q$  is a nonnegative constant. If  $\int (1 \wedge |x|) \mu(dx) < \infty$ , we write

$$\psi(z) = ia'z + \frac{1}{2}Qz^2 + \int_{\mathbf{R}} (1 - e^{i\langle z, x \rangle}) \mu(dx).$$

## 2.1 Motivation

Let us start by recalling a beautiful result of Bretagnolle [3]. Define

$$\mathcal{C} = \{x \in \mathbf{R} : P\{X_t = x \text{ for some } t > 0\} > 0\}, \quad (2.1)$$

and consider the following different cases:

- A.  $Q > 0$ .
- B.  $Q = 0$ ;  $\int (1 \wedge |x|)\mu(dx) = +\infty$ .
- C.  $Q = 0$ ;  $\int (1 \wedge |x|)\mu(dx) < +\infty$ . We further decompose it into the following three subcases:
  - $C_1$ .  $a' = 0$ ,
  - $C_2$ .  $a' > 0$ ,  $\mu$  does not charge  $\mathbf{R}^- := \{x \in \mathbf{R} : x < 0\}$ .
  - $C_3$ .  $a' > 0$ ,  $\mu$  charges  $\mathbf{R}^-$ .

**Theorem 2.1** (*Bretagnolle [3, Theorem 8]*)

- (i) For Case A,  $\mathcal{C} = \mathbf{R}$  and 0 is a regular point of  $\{0\}$ .
- (ii) For Case B, either  $\mathcal{C} = \emptyset$  or  $\mathcal{C} = \mathbf{R}$ , and if  $\mathcal{C} = \mathbf{R}$  then 0 is a regular point of  $\{0\}$ .
- (iii) For Case C, suppose that  $X$  is not a compound Poisson process, then
  - (a) for Case  $C_1$ ,  $\mathcal{C} = \emptyset$ ;
  - (b) for Case  $C_2$ ,  $\mathcal{C} = \mathbf{R}^+ := \{x \in \mathbf{R} : x > 0\}$  and 0 is not a regular point of  $\{0\}$ ;
  - (c) for Case  $C_3$ ,  $\mathcal{C} = \mathbf{R}$  and 0 is not a regular point of  $\{0\}$ .

For Case A, and Case B with  $\mathcal{C} = \mathbf{R}$ , only the empty set is a semipolar set. Hence (H) holds for these two cases. For Case  $C_2$  and Case  $C_3$ , any singleton  $\{x\}$  is semipolar but non-polar. Thus (H) doesn't hold for these two cases. Therefore, for one-dimensional Lévy processes, we need only consider whether (H) holds for Case B with  $\mathcal{C} = \emptyset$  and Case  $C_1$ .

For Case B, Kesten [12, Theorem 1(f)] tells us that if  $\int_0^\infty (1 \wedge x)\mu(dx) < \infty$  or  $\int_{-\infty}^0 (1 \wedge |x|)\mu(dx) < \infty$ , then  $\mathcal{C} = \mathbf{R}$ . Thus, any  $x \in \mathbf{R}$  is a regular point of  $\{x\}$  and hence (H) holds for this case. As a consequence, any spectrally one sided one-dimensional Lévy process with unbounded variation satisfies (H). Therefore, for Case B, we need only consider the case that both  $\int_0^\infty (1 \wedge x)\mu(dx) = \infty$  and  $\int_{-\infty}^0 (1 \wedge |x|)\mu(dx) = \infty$ .

Denote by  $\mu_+$  and  $\mu_-$  the restriction of the Lévy measure  $\mu$  on  $(0, \infty)$  and  $(-\infty, 0)$ , respectively. Let  $X_1$  and  $X_2$  be two independent Lévy processes with Lévy measures  $\mu_+$  and  $\mu_-$ , respectively. For Case B with  $\int_0^\infty (1 \wedge x)\mu(dx) = \infty$  and  $\int_{-\infty}^0 (1 \wedge |x|)\mu(dx) = \infty$ , both  $X_1$  and  $X_2$  belong to Case B with  $\mathcal{C} = \mathbf{R}$  and hence satisfy (H). Obviously,  $X$  can be regarded as the sum of  $X_1$  and  $X_2$ . This observation provides a motivation for us to consider (H) for the sum of two independent Lévy processes.

## 2.2 Main results

First, we present a result which extends [12, Theorem 1(f)]. Let  $\mu$  be the Lévy measure. We denote by  $\bar{\mu}_-$  the image measure of  $\mu_-$  under the map

$$x \mapsto -x, \quad \forall x \in (-\infty, 0).$$

**Theorem 2.2** *Suppose that  $Q = 0$  and  $\int_0^\infty (1 \wedge x)\mu_+(dx) = \infty$ . If there exist  $\delta \in (0, 1), k \in [0, 1)$ , and a measure  $\nu$  on  $\mathbf{R}^+$  satisfying  $\int_{(0, \delta)} x\nu(dx) < \infty$ , such that*

$$\bar{\mu}_- \leq k\mu_+ + \nu. \quad (2.2)$$

*Then  $X$  satisfies (H).*

**Proof.** We assume without loss of generality that  $k > 0$ . Define  $\mu_2$  to be the symmetric measure on  $\mathbf{R} \setminus \{0\}$  satisfying  $\mu_2 = (\bar{\mu}_- - \nu)^+$  on  $(0, \delta)$  and  $\mu_2 = 0$  on  $[\delta, \infty)$ , where  $(\bar{\mu}_- - \nu)^+$  denotes the positive part of the signed measure  $\bar{\mu}_- - \nu$ . Denote  $\mu_1 = \mu - \mu_2$ . Let  $X_1$  and  $X_2$  be two independent one-dimensional Lévy processes with Lévy-Khintchine exponents  $(a, 0, \mu_1)$  and  $(0, 0, \mu_2)$ , respectively. Since  $X$  and  $X_1 + X_2$  have the same law, to show that  $X$  satisfies (H), it is sufficient to show that  $X_1 + X_2$  satisfies (H). We denote by  $\psi_1$  and  $\psi_2$  the Lévy-Khintchine exponents of  $X_1$  and  $X_2$ , respectively.

By (2.2), we get

$$\int_0^\infty (1 \wedge x)\mu_1(dx) \geq \int_{(0, \delta)} x\mu_1(dx) \geq (1 - k) \int_{(0, \delta)} x\mu_+(dx) = \infty,$$

and

$$\begin{aligned} \int_{-\infty}^0 (1 \wedge |x|)\mu_1(dx) &= \int_{(-\infty, -\delta]} (1 \wedge |x|)\mu_-(dx) + \int_{(-\delta, 0)} |x|\mu_1(dx) \\ &\leq \mu_-((-\infty, -\delta]) + \int_{(0, \delta)} x\nu(dx) \\ &< \infty. \end{aligned}$$

Then, we obtain by [12, Theorem 1(f)] that  $X_1$  belongs to Case B with  $\mathcal{C} = \mathbf{R}$ . Therefore, we obtain by [12] that

$$\int_0^\infty \operatorname{Re}([1 + \psi_1(z)]^{-1})dz < \infty. \quad (2.3)$$

By (2.2) and the definition of  $\psi_2$ , we obtain that for  $z \in \mathbf{R}$ ,

$$\begin{aligned} \psi_2(z) = \operatorname{Re}\psi_2(z) &= 2 \int_{(0, \delta)} (1 - \cos(zx))\mu_2(dx) \\ &\leq 2k \int_{(0, \delta)} (1 - \cos(zx))\mu_+(dx) \\ &\leq \frac{2k}{1 - k} \int_{(0, \delta)} (1 - \cos(zx))\mu_1(dx) \\ &\leq \frac{2k}{1 - k} \operatorname{Re}\psi_1(z). \end{aligned} \quad (2.4)$$

By (2.3) and (2.4), we get

$$\begin{aligned}
& \int_0^\infty \operatorname{Re}([1 + \psi_1(z) + \psi_2(z)]^{-1}) dz \\
&= \int_0^\infty \frac{1}{1 + \operatorname{Re}\psi_1(z) + \operatorname{Re}\psi_2(z) + \frac{(\operatorname{Im}\psi_1(z))^2}{1 + \operatorname{Re}\psi_1(z) + \operatorname{Re}\psi_2(z)}} dz \\
&\leq \int_0^\infty \frac{1}{1 + \operatorname{Re}\psi_1(z) + \frac{(\operatorname{Im}\psi_1(z))^2}{1 + (\frac{1+k}{1-k})\operatorname{Re}\psi_1(z)}} dz \\
&\leq \frac{1+k}{1-k} \int_0^\infty \operatorname{Re}([1 + \psi_1(z)]^{-1}) dz \\
&< \infty.
\end{aligned}$$

Then, we obtain by [12] that any singleton is non-polar for  $X_1 + X_2$ . Hence any point  $x \in \mathbf{R}$  is a regular point of  $\{x\}$  by Theorem 2.1(ii). Therefore,  $X_1 + X_2$  satisfies (H).  $\square$

We now give a novel condition on the Lévy measure  $\mu$  which implies (H) for a large class of one-dimensional Lévy processes.

**Theorem 2.3** *If*

$$\liminf_{\varepsilon \downarrow 0} \frac{\int_{-\varepsilon}^{\varepsilon} x^2 \mu(dx)}{\varepsilon / |\log \varepsilon|} > 0, \tag{2.5}$$

*then  $X$  satisfies (H).*

Note that, different from most sufficient conditions given in the diagram of Section 1, condition (2.5) does not require any controllability of  $\operatorname{Im}(\psi)$  by  $\operatorname{Re}(\psi)$ . Before proving Theorem 2.3, we give a necessary and sufficient condition for the validity of (H) for general Lévy processes.

**Proposition 2.4** *Suppose that  $X$  is a Lévy process on  $\mathbf{R}^n$  which has resolvent densities w.r.t. the Lebesgue measure. Let  $f$  be a positive increasing function on  $[1, \infty)$  such that  $\int_N^\infty (\lambda f(\lambda))^{-1} d\lambda = \infty$  for some  $N \geq 1$ . Then (H) holds for  $X$  if and only if*

$$\lim_{\lambda \rightarrow \infty} \int_{\{B(z) > A(z)f(A(z))\}} \frac{\lambda}{\lambda^2 + B^2(z)} |\hat{\nu}(z)|^2 dz = 0$$

*for any finite measure  $\nu$  of finite 1-energy.*

**Proof.** This is a direct consequence of [10, Theorems 4.3 and 5.1].  $\square$

**Proof of Theorem 2.3.** By (2.5), we know that there exist constants  $N_1$  and  $c$  satisfying  $N_1 > 1$  and  $0 < c < 1$  such that

$$\int_{-\frac{1}{|z|}}^{\frac{1}{|z|}} x^2 \mu(dx) \geq \frac{c}{|z| \log |z|}, \quad \text{if } |z| \geq N_1.$$

Note that  $1 - \cos x \geq \frac{x^2}{4}$  when  $|x| \leq 1$ . Then, for  $|z| \geq N_1$ , we have

$$\begin{aligned}
\operatorname{Re} \psi(z) &= \int_{\mathbf{R}} (1 - \cos(zx)) \mu(dx) \\
&\geq \int_{-\frac{1}{|z|}}^{\frac{1}{|z|}} (1 - \cos(zx)) \mu(dx) \\
&\geq \frac{z^2}{4} \int_{-\frac{1}{|z|}}^{\frac{1}{|z|}} x^2 \mu(dx) \\
&\geq \frac{c|z|}{4 \log |z|}.
\end{aligned} \tag{2.6}$$

We define  $f(\lambda) = \frac{4}{c} \log(\frac{4\lambda}{c}) [\log \log(\frac{4\lambda}{c})]$  for  $\lambda > c$ . Then,  $f(\lambda)$  is a positive increasing function on  $(c, \infty)$  and satisfy

$$\int_c^\infty \frac{1}{\lambda f(\lambda)} d\lambda = \int_4^\infty \frac{1}{u \log u [\log \log u]} du = \infty.$$

We fix a constant  $\alpha$  satisfying  $0 < \alpha < 1$ . By  $\lim_{z \rightarrow \infty} \frac{z/\log z}{z^\alpha} = +\infty$ , we know that there exists a constant  $N_2 > 0$  such that

$$\frac{z}{\log z} \geq z^\alpha, \quad \forall z \geq N_2. \tag{2.7}$$

We define  $g(z) = \log \log(\frac{z}{\log z})$  for  $z > e$ . It is easy to see that  $g(z)$  is an increasing positive function on  $(e, \infty)$ .

By (2.6) and (2.7), we obtain that for any  $N_0 > \max\{N_1, N_2, e\}$ ,

$$\begin{aligned}
&\limsup_{\lambda \rightarrow \infty} \int_{\{B(z) > A(z)f(A(z))\}} \frac{\lambda}{\lambda^2 + B^2(z)} dz \\
&= \limsup_{\lambda \rightarrow \infty} \int_{\{B(z) > A(z)f(A(z)), |z| > N_0\}} \frac{\lambda}{\lambda^2 + B^2(z)} dz \\
&\leq \limsup_{\lambda \rightarrow \infty} \int_{\{|z| > N_0\}} \frac{\lambda}{\lambda^2 + \frac{z^2}{\log^2 |z|} \log^2(\frac{|z|}{\log |z|}) [\log \log(\frac{|z|}{\log |z|})]^2} dz \\
&\leq \limsup_{\lambda \rightarrow \infty} \int_{\{|z| > N_0\}} \frac{\lambda}{\lambda^2 + |z|^2 \left(\frac{\log(|z|^\alpha)}{\log |z|}\right)^2 g^2(|z|)} dz \\
&\leq \lim_{\lambda \rightarrow \infty} \int_{\{|z| > N_0\}} \frac{\lambda}{\lambda^2 + \alpha^2 g^2(N_0) |z|^2} dz \\
&= \limsup_{\lambda \rightarrow \infty} \frac{2}{\alpha g(N_0)} \int_{\alpha N_0 g(N_0)}^\infty \frac{\lambda}{\lambda^2 + u^2} du \\
&\leq \frac{\pi}{\alpha g(N_0)}.
\end{aligned}$$



Since  $\lim_{N_0 \rightarrow \infty} g(N_0) = 0$ , we obtain

$$\lim_{\lambda \rightarrow \infty} \int_{\{B(z) > A(z)f(A(z))\}} \frac{\lambda}{\lambda^2 + B^2(z)} dz = 0. \quad (2.8)$$

By (2.6) and [7], we know that  $X$  has bounded continuous transition densities. Therefore,  $X$  satisfies (H) by (2.8) and Proposition 2.4.  $\square$

**Remark 2.5** For  $\alpha > 0$ , we define the measure  $\nu_\alpha$  on  $(-1, 1)$  by

$$\nu_\alpha(dx) := |x \log |x||^{1+\alpha} \mu(dx), \quad x \in (-1, 1).$$

We remark that our condition (2.5) only requires slightly more than  $\nu_\alpha$  is an infinite measures on  $(-1, 1)$  for any  $\alpha > 0$ .

(i) Condition (2.5) implies that any  $\nu_\alpha$  is an infinite measure on  $(-1, 1)$ . In fact, by (2.5), we get

$$\lim_{\varepsilon \downarrow 0} \frac{|\log \varepsilon|^{1+\alpha} \int_{-\varepsilon}^{\varepsilon} x^2 \mu(dx)}{\varepsilon} = \infty. \quad (2.9)$$

If  $\nu_\alpha$  is a finite measure on  $(-1, 1)$ , then

$$\begin{aligned} & \limsup_{\varepsilon \downarrow 0} \frac{|\log \varepsilon|^{1+\alpha} \int_{-\varepsilon}^{\varepsilon} x^2 \mu(dx)}{\varepsilon} \\ & \leq \limsup_{\varepsilon \downarrow 0} \frac{\int_{-\varepsilon}^{\varepsilon} x^2 |\log |x||^{1+\alpha} \mu(dx)}{\varepsilon} \\ & = \limsup_{\varepsilon \downarrow 0} \frac{\int_{-\varepsilon}^{\varepsilon} |x| \nu_\alpha(dx)}{\varepsilon} \\ & \leq \nu_\alpha(-1, 1), \end{aligned}$$

which contradicts (2.9).

(ii) If for some  $\beta > 2$ ,

$$\liminf_{\varepsilon \downarrow 0} \frac{\int_{-\varepsilon}^{\varepsilon} x^2 \mu(dx)}{\varepsilon / |\log \varepsilon|^\beta} = 0, \quad (2.10)$$

then  $\nu_\alpha$  is a finite measure on  $(-1, 1)$  for any  $\alpha \in (0, \beta - 2)$ .

We only prove  $\nu_\alpha(0, 1) < \infty$ . The proof that  $\nu_\alpha(-1, 0) < \infty$  is similar so we omit it. By (2.10), we know that there exist constants  $c$  and  $\delta$  satisfying  $c > 0$  and  $0 < \delta < 1$  such that

$$\int_0^\varepsilon x^2 \mu(dx) \leq \frac{c\varepsilon}{|\log \varepsilon|^\beta}, \quad \forall \varepsilon \in (0, \delta).$$

Note that  $f(x) = x/|\log x|^{1+\alpha}$  is an increasing function on  $(0, 1)$ . Then, for any  $\varepsilon \in (0, \delta)$ , we have

$$\frac{\varepsilon/2}{|\log(\varepsilon/2)|^{1+\alpha}} \nu([\varepsilon/2, \varepsilon]) \leq \int_{\varepsilon/2}^\varepsilon \frac{x}{|\log x|^{1+\alpha}} \nu(dx) \leq \int_0^\varepsilon x^2 \mu(dx) \leq \frac{c\varepsilon}{|\log \varepsilon|^\beta},$$

which implies that

$$\nu([\varepsilon/2, \varepsilon]) \leq \frac{2c|\log(\varepsilon/2)|^{1+\alpha}}{|\log \varepsilon|^\beta}.$$

We fix a  $K \in \mathbf{N}$  satisfying  $\frac{1}{2^K} < \delta$ . Then,

$$\begin{aligned} \nu(0, 1) &= \sum_{n=1}^K \nu([1/2^n, 1/2^{(n-1)})) + \sum_{n=K+1}^{\infty} \nu([1/2^n, 1/2^{(n-1)})) \\ &\leq \sum_{n=1}^K \nu([1/2^n, 1/2^{(n-1)})) + \sum_{n=K+1}^{\infty} \frac{2c|\log(1/2^n)|^{1+\alpha}}{|\log(1/2^{(n-1)})|^\beta} \\ &= \sum_{n=1}^K \nu([1/2^{(n-1)}, 1/2^n)) + 2c \sum_{n=K+1}^{\infty} \frac{(n \log 2)^{1+\alpha}}{((n-1) \log 2)^\beta} \\ &< \infty. \end{aligned}$$

From the proof of Theorem 2.3, we can see that the following result extending [10, Theorem 4.7] holds.

**Proposition 2.6** *If*

$$\liminf_{|z| \rightarrow \infty} \frac{\operatorname{Re} \psi(z)}{|z|/\log |z|} > 0,$$

*then  $X$  satisfies (H).*

Following the proof of Theorem 2.3, we can also prove the following proposition.

**Proposition 2.7** *If*

$$\liminf_{\varepsilon \rightarrow 0} \frac{\int_{-\varepsilon}^{\varepsilon} x^2 \mu(dx)}{\frac{\varepsilon}{|\log \varepsilon| |\log |\log \varepsilon||}}} > 0,$$

*then  $X$  satisfies (H).*

## 2.3 An example

We give an application of Theorem 2.3. Note that in the following example, there is no assumption on  $a$  or  $Q$ .

**Example 2.8** *Let  $X$  be a Lévy process on  $\mathbf{R}$  with Lévy measure  $\mu$ . Suppose that there exist positive constants  $c, \delta$ , and a finite measure  $\nu$  on  $(0, \delta)$  such that*

$$\mu(dx) + \nu(dx) \geq \frac{c}{x^2 |\log x|} dx \quad \text{on } (0, \delta).$$

Then  $X$  satisfies (H).

In fact, we have

$$\begin{aligned}
\liminf_{\varepsilon \downarrow 0} \frac{\int_0^\varepsilon x^2 \cdot \frac{c}{x^2 |\log x|} dx}{\varepsilon / |\log \varepsilon|} &\geq \liminf_{\varepsilon \downarrow 0} \frac{\int_{\varepsilon/2}^\varepsilon \frac{c}{|\log x|} dx}{\varepsilon / |\log \varepsilon|} \\
&\geq \liminf_{\varepsilon \downarrow 0} \frac{\frac{c}{|\log \varepsilon|} \cdot \frac{\varepsilon}{2}}{\varepsilon / |\log \varepsilon|} \\
&= \frac{c}{2},
\end{aligned}$$

and

$$\limsup_{\varepsilon \downarrow 0} \frac{\int_0^\varepsilon x^2 \nu(dx)}{\varepsilon / |\log \varepsilon|} \leq \limsup_{\varepsilon \downarrow 0} \frac{\varepsilon^2 \nu(0, 1)}{\varepsilon / |\log \varepsilon|} = 0.$$

Then (2.5) holds and therefore  $X$  satisfies (H) by Theorem 2.3.

### 3 (H) for sum of Lévy processes: no assumption on resolvent densities

From now on till the end of the paper, we consider Hunt's hypothesis (H) for general  $\mathbf{R}^n$ -valued Lévy processes. In this section, we discuss (H) for the sum of two independent Lévy processes without any assumption on resolvent densities. In the next section, we discuss (H) for the sum of two independent Lévy processes under the assumption that resolvent densities exist.

#### 3.1 Main results

**Theorem 3.1** *Let  $X_1$  and  $X_2$  be two independent Lévy processes on  $\mathbf{R}^n$ . If  $X_1$  satisfies (H) and  $X_2$  is a compound Poisson process, then  $X_1 + X_2$  satisfies (H).*

**Theorem 3.2** *Let  $X_1$  and  $X_2$  be two independent Lévy processes on  $\mathbf{R}^n$ . If both  $X_1$  and  $X_2$  satisfy condition (S), then  $X_1 + X_2$  satisfies (H).*

As a direct consequence of Theorem 3.1, we can strengthen [10, Theorem 2.1] as follows:

**Proposition 3.3** *Let  $X$  be a Lévy process on  $\mathbf{R}^n$  with Lévy-Khintchine exponent  $(a, Q, \mu)$ . Suppose that  $\mu_1$  is a finite measure on  $\mathbf{R}^n \setminus \{0\}$  such that  $\mu_1 \leq \mu$ . Denote  $\mu_2 := \mu - \mu_1$  and let  $X'$  be a Lévy process on  $\mathbf{R}^n$  with Lévy-Khintchine exponent  $(a', Q, \mu_2)$ , where  $a' := a + \int_{\{|x| < 1\}} x \mu_1(dx)$ . Then*

(i)  $X$  and  $X'$  have same semipolar sets.

(ii)  $X$  and  $X'$  have same essentially polar sets.

(iii) if  $X'$  satisfies (H), then  $X$  satisfies (H).

(iv) if  $X$  satisfies (H) and  $X'$  has resolvent densities w.r.t. the Lebesgue measure, then  $X'$  satisfies (H).

### 3.2 Proof of Theorem 3.1

Before proving Theorem 3.1, we present some lemmas, which have their own interests.

**Lemma 3.4** *Let  $X$  be a Lévy process on  $\mathbf{R}^n$  ( $n > 1$ ) satisfying (H). Then, for any nonempty proper subspace  $S$  of  $\mathbf{R}^n$ , the projection process  $Y$  of  $X$  on  $S$  satisfies (H).*

**Proof.** By virtue of the orthogonal transformation (cf. [8, Section 2.2]), we can assume without loss of generality that  $S = \{(x_1, \dots, x_n) \in \mathbf{R}^n | x_{k+1} = \dots = x_n = 0\}$  for some integer  $k, 1 \leq k < n$ . Then, the projection process  $Y$  of  $X$  can be regarded as a Lévy process on  $\mathbf{R}^k$ . Let  $C \subset \mathbf{R}^k$  be a semipolar set for  $Y$ . We define

$$D = \{(x_1, \dots, x_n) \in \mathbf{R}^n | (x_1, \dots, x_k) \in C\}.$$

By the definition of semipolar set, we find that  $D$  is a semipolar set for  $X$ . Further, by the assumption that  $X$  satisfies (H), we conclude that  $D$  is a polar set for  $X$ . Therefore, as the projection of  $D$  on  $S$ ,  $C$  is a polar set for  $Y$ .  $\square$

**Lemma 3.5** *Let  $X$  be a Lévy process on  $\mathbf{R}^n$  ( $n > 1$ ) with Lévy-Khintchine exponent  $(a, Q, \mu)$ . Suppose that for some proper subspace  $S$  of  $\mathbf{R}^n$ , the projection process  $X_S$  of  $X$  on  $S$  satisfies (H) and  $\mu(\mathbf{R}^n \setminus S) < \infty$ . Then  $X$  satisfies (H).*

**Proof.** By virtue of the orthogonal transformation, we can assume without loss of generality that  $S = \{(x_1, \dots, x_n) \in \mathbf{R}^n | x_{k+1} = \dots = x_n = 0\}$  for some integer  $k, 1 \leq k < n$ . By the Lévy-Itô decomposition (cf. the proof of [8, Theorem 1.2]), we may express  $X$  as

$$X = X^{(1)} + X^{(2)},$$

where  $X^{(1)} = (X_S, 0)$  can be regarded as a  $k$ -dimensional Lévy process on  $\mathbf{R}^k \times \{0\}$  which satisfies (H), and  $X^{(2)}$  is a compound Poisson process on  $\mathbf{R}^n$  which is independent of  $X^{(1)}$ . Then, by following the proof of (ii)  $\Rightarrow$  (i) of [8, Theorem 1.2], we conclude that  $X$  satisfies (H).  $\square$

**Lemma 3.6** *Let  $X_1$  and  $X_2$  be two independent Lévy processes on  $\mathbf{R}^m$  and  $\mathbf{R}^n$ , respectively. If  $X_1$  satisfies (H) and  $X_2$  is a compound Poisson process, then  $X = (X_1, X_2)$  satisfies (H).*

**Proof.** This is a direct consequence of Lemma 3.5.  $\square$

**Proof of Theorem 3.1.** By Lemma 3.6, we find that the  $\mathbf{R}^{2n}$ -valued Lévy process  $(X_1, X_2)$  satisfies (H). Further, by the orthogonal transformation, we find that the Lévy process  $\frac{1}{\sqrt{2}}(X_1 + X_2, X_2 - X_1)$  satisfies (H). Therefore,  $X_1 + X_2$  satisfies (H) by Lemma 3.4.  $\square$

### 3.3 Proof of Theorem 3.2

Before giving the proof for Theorem 3.2, we prove the following lemma.

**Lemma 3.7** *Let  $M$  be a symmetric nonnegative definite  $n \times n$  matrix. Then,  $x \in \sqrt{M}\mathbf{R}^n$  if and only if there exists a constant  $c > 0$  such that*

$$|\langle x, z \rangle| \leq c\sqrt{\langle z, Mz \rangle}, \quad \forall z \in \mathbf{R}^n. \quad (3.1)$$

**Proof.** Suppose that  $x \in \sqrt{M}\mathbf{R}^n$ . Then, there exists a  $y \in \mathbf{R}^n$  such that  $x = \sqrt{M}y$  and thus

$$\begin{aligned} |\langle x, z \rangle| &= |\langle \sqrt{M}y, z \rangle| \\ &= |\langle y, \sqrt{M}z \rangle| \\ &\leq \sqrt{\langle y, y \rangle} \sqrt{\langle \sqrt{M}z, \sqrt{M}z \rangle} \\ &= \sqrt{\langle y, y \rangle} \sqrt{\langle z, Mz \rangle}. \end{aligned}$$

Therefore, (3.1) holds with  $c = 1 + \sqrt{\langle y, y \rangle}$ .

Now we suppose that (3.1) holds. Denote by  $k$  the rank of  $M$ . If  $k = n$  or  $0$ , it is easy to see that  $x \in \sqrt{M}\mathbf{R}^n$ . Hence we may assume that  $n \geq 2$  and  $1 \leq k \leq n - 1$ . Since  $M$  is a symmetric nonnegative definite  $n \times n$  matrix, there exists an orthogonal matrix  $O$  such that

$$OMO^T = \text{diag}(\lambda_1, \dots, \lambda_n) := F,$$

where  $\lambda_1 \geq \dots \geq \lambda_k > 0$ ,  $\lambda_i = 0$  for  $i = k + 1, \dots, n$ , and  $O^T$  denotes the transpose of  $O$ . We can rewrite (3.1) as follows:

$$|\langle Ox, Oz \rangle| \leq c\sqrt{\langle Oz, F(Oz) \rangle}, \quad \forall z \in \mathbf{R}^n,$$

equivalently,

$$|\langle Ox, z' \rangle| \leq c\sqrt{\langle z', Fz' \rangle}, \quad \forall z' \in \mathbf{R}^n. \quad (3.2)$$

We claim that  $Ox \in \sqrt{F}\mathbf{R}^n = \mathbf{R}^k \times \{0\}$ . Let  $Ox = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ . If  $Ox \notin \mathbf{R}^k \times \{0\}$ , then there exists  $j \in \{k + 1, \dots, n\}$  such that  $\bar{x}_j \neq 0$ . Let  $z' = (z'_1, \dots, z'_n)$  with  $z'_j = 1$  and  $z'_i = 0$  for  $i \neq j$ . Thus, we obtain by (3.2) that

$$0 < |\bar{x}_j| = |\langle Ox, z' \rangle| \leq c\sqrt{\langle z', Fz' \rangle} = 0.$$

This is a contradiction and hence  $Ox \in \sqrt{F}\mathbf{R}^n$ . Therefore,  $x \in \sqrt{M}\mathbf{R}^n$ . □

**Proof of Theorem 3.2.** We denote the Lévy-Khintchine exponents of  $X_1$  and  $X_2$  by  $(a_1, Q_1, \mu_1)$  and  $(a_2, Q_2, \mu_2)$ , respectively. By Lemma 3.7, we find that  $\sqrt{Q_1}\mathbf{R}^n \subset \sqrt{Q_1 + Q_2}\mathbf{R}^n$  and  $\sqrt{Q_2}\mathbf{R}^n \subset \sqrt{Q_1 + Q_2}\mathbf{R}^n$ . Thus

$$(\mu_1 + \mu_2)(\mathbf{R}^n \setminus \sqrt{Q_1 + Q_2}\mathbf{R}^n) < \infty. \quad (3.3)$$

By [8, Theorem 1.2], we know that both  $X_1$  and  $X_2$  satisfy the Kanda-Forst condition and hence  $X_1 + X_2$  satisfies the Kanda-Forst condition. Therefore,  $X_1 + X_2$  satisfies (H) by (3.3) and [8, Theorem 1.2]. □

## 4 (H) for sum of Lévy processes under assumption that resolvent densities exist

Throughout this section, we assume that  $X_1$  and  $X_2$  are two independent Lévy processes on  $\mathbf{R}^n$  such that  $X_1 + X_2$  has resolvent densities w.r.t. the Lebesgue measure. We denote by  $\psi_1$  and  $\psi_2$  the Lévy-Khintchine exponents of  $X_1$  and  $X_2$ , respectively.

### 4.1 Main results

**Theorem 4.1** *Suppose that*

- (i)  $X_1$  has resolvent densities w.r.t. the Lebesgue measure and satisfies (H).
- (ii) Any finite measure  $\nu$  of finite 1-energy w.r.t.  $X_1 + X_2$  has finite 1-energy w.r.t.  $X_1$ .
- (iii) There exists a constant  $c > 0$  such that

$$|\operatorname{Im}\psi_2| \leq c(1 + \operatorname{Re}\psi_1 + \operatorname{Re}\psi_2).$$

Then  $X_1 + X_2$  satisfies (H).

**Proposition 4.2** *If one of the following conditions is fulfilled, then any finite measure  $\nu$  of finite 1-energy w.r.t.  $X_1 + X_2$  has finite 1-energy w.r.t.  $X_1$ .*

- (i) There exists a constant  $c > 0$  such that

$$|\psi_2| \leq c(1 + \operatorname{Re}(\psi_1)).$$

- (ii) There exists a constant  $c > 0$  such that

$$\begin{cases} \operatorname{Re}\psi_2 \leq c \left( 1 + \operatorname{Re}\psi_1 + \frac{(\operatorname{Im}\psi_1)^2}{1 + \operatorname{Re}\psi_1} \right), \\ |\operatorname{Im}\psi_2| \leq c(1 + \operatorname{Re}\psi_1 + \operatorname{Re}\psi_2). \end{cases}$$

- (iii) There exists a constant  $c > 0$  such that

$$\begin{cases} \operatorname{Re}\psi_2 \leq c \left( 1 + \operatorname{Re}\psi_1 + \frac{(\operatorname{Im}\psi_1)^2}{1 + \operatorname{Re}\psi_1} \right), \\ (\operatorname{Im}\psi_2)^2 \leq c(1 + \operatorname{Re}\psi_1 + \operatorname{Re}\psi_2) \left( 1 + \operatorname{Re}\psi_1 + \frac{(\operatorname{Im}\psi_1)^2}{1 + \operatorname{Re}\psi_1} \right). \end{cases} \quad (4.1)$$

**Corollary 4.3** *Suppose that*

- (i)  $X_1$  has bounded resolvent densities w.r.t. the Lebesgue measure and satisfies (H).
- (ii) There exists a constant  $c > 0$  such that

$$|\operatorname{Im}\psi_2| \leq c(1 + \operatorname{Re}\psi_1 + \operatorname{Re}\psi_2).$$

Then  $X_1 + X_2$  satisfies (H).

**Remark 4.4** Let  $X$  be a one-dimensional Lévy process and the set  $\mathcal{C}$  be defined as in (2.1). By [14, Theorem 43.21, Case 5], we know that if  $X$  belongs to Case B (defined as in Section 2) with  $\mathcal{C} = \mathbf{R}$ , then  $X$  has bounded resolvent densities w.r.t the Lebesgue measure. In particular,

(i) the one-dimensional Brownian motion has bounded resolvent densities.

(ii) any spectrally one sided one-dimensional Lévy process with unbounded variation has bounded resolvent densities.

(iii) any one-dimensional Lévy process satisfying the conditions of Theorem 2.2 has bounded resolvent densities.

**Proposition 4.5** Let  $f$  be a positive increasing function on  $[1, \infty)$  such that  $\int_N^\infty (\lambda f(\lambda))^{-1} d\lambda = \infty$  for some  $N \geq 1$ . Suppose that

(i) There are two measurable functions  $\phi_{11}$  and  $\phi_{12}$  on  $\mathbf{R}^n$  such that  $\text{Im}\psi_1 = \phi_{11} + \phi_{12}$ , and

$$|\phi_{11}| \leq (1 + \text{Re}\psi_1)f(1 + \text{Re}\psi_1), \quad \int_{\mathbf{R}^n} \frac{|\phi_{12}(z)|}{|1 + \psi_1(z)|^2} dz < \infty.$$

(ii)

$$|\text{Im}\psi_2| \leq (1 + \text{Re}\psi_1 + \text{Re}\psi_2)f(1 + \text{Re}\psi_1 + \text{Re}\psi_2).$$

Then  $X_1 + X_2$  satisfies (H).

## 4.2 Proofs

Before giving the proof for Theorem 4.1, we prove the following lemma.

**Lemma 4.6** Suppose that there exists a constant  $c > 0$  such that

$$|\text{Im}\psi_2| \leq c(1 + \text{Re}\psi_1 + \text{Re}\psi_2). \quad (4.2)$$

Then, there exists a constant  $\gamma > 0$  such that

$$|1 + \psi_1 + \psi_2|^2 \geq \gamma|1 + \psi_1|^2.$$

**Proof.** Suppose that (4.2) holds. We take  $\gamma \in (0, \frac{1}{4})$  such that  $(1 - \gamma)(1 + \frac{1}{4c^2}) > 1$ . Then, for any  $x \in \mathbf{R}$ , we have

$$\begin{aligned} (x+1)^2 - \left(\gamma x^2 - \frac{1}{4c^2}\right) &= (1-\gamma)x^2 + 2x + \left(1 + \frac{1}{4c^2}\right) \\ &= (1-\gamma)\left(x + \frac{1}{1-\gamma}\right)^2 + \frac{1}{1-\gamma}\left((1-\gamma)\left(1 + \frac{1}{4c^2}\right) - 1\right) \\ &\geq \frac{1}{1-\gamma}\left((1-\gamma)\left(1 + \frac{1}{4c^2}\right) - 1\right) \\ &> 0, \end{aligned}$$

which implies that

$$(x+1)^2 > \gamma x^2 - \frac{1}{4c^2}, \quad \forall x \in \mathbf{R}. \quad (4.3)$$

By (4.3), we get

$$(\operatorname{Im}\psi_1 + \operatorname{Im}\psi_2)^2 \geq \gamma(\operatorname{Im}\psi_1)^2 - \frac{1}{4c^2}(\operatorname{Im}\psi_2)^2. \quad (4.4)$$

Therefore, we obtain by (4.2) and (4.4) that

$$\begin{aligned} |1 + \psi_1 + \psi_2|^2 &= (1 + \operatorname{Re}\psi_1 + \operatorname{Re}\psi_2)^2 + (\operatorname{Im}\psi_1 + \operatorname{Im}\psi_2)^2 \\ &= \left[ \left( \frac{1}{2} + \frac{1}{2}\operatorname{Re}\psi_1 \right) + \left( \frac{1}{2} + \frac{1}{2}\operatorname{Re}\psi_1 + \operatorname{Re}\psi_2 \right) \right]^2 + (\operatorname{Im}\psi_1 + \operatorname{Im}\psi_2)^2 \\ &\geq \left( \frac{1}{2} + \frac{1}{2}\operatorname{Re}\psi_1 \right)^2 + \left( \frac{1}{2} + \frac{1}{2}\operatorname{Re}\psi_1 + \operatorname{Re}\psi_2 \right)^2 + \gamma(\operatorname{Im}\psi_1)^2 - \frac{1}{4c^2}(\operatorname{Im}\psi_2)^2 \\ &\geq \frac{1}{4}(1 + \operatorname{Re}\psi_1)^2 + \frac{1}{4}(1 + \operatorname{Re}\psi_1 + \operatorname{Re}\psi_2)^2 + \gamma(\operatorname{Im}\psi_1)^2 \\ &\quad - \frac{1}{4c^2} \cdot c^2(1 + \operatorname{Re}\psi_1 + \operatorname{Re}\psi_2)^2 \\ &= \frac{1}{4}(1 + \operatorname{Re}\psi_1)^2 + \gamma(\operatorname{Im}\psi_1)^2 \\ &> \gamma[(1 + \operatorname{Re}\psi_1)^2 + (\operatorname{Im}\psi_1)^2] \\ &= \gamma|1 + \psi_1|^2. \end{aligned}$$

The proof is complete.  $\square$

**Proof of Theorem 4.1.** Let  $\nu$  be a finite measure of finite 1-energy w.r.t.  $X_1 + X_2$ . By Assumption (ii),  $\nu$  has finite 1-energy w.r.t.  $X_1$ . Then, by Assumption (i) and [9, Proposition 2.2], we get

$$\lim_{\lambda \rightarrow \infty} \int_{\mathbf{R}^n} \frac{\lambda}{\lambda^2 + |1 + \psi_1(z)|^2} |\hat{\nu}(z)|^2 dz = 0. \quad (4.5)$$

By Assumption (iii) and Lemma 4.6, we find that there exists a constant  $\gamma > 0$  such that

$$|1 + \psi_1 + \psi_2|^2 \geq \gamma|1 + \psi_1|^2. \quad (4.6)$$

By (4.5) and (4.6), we obtain that

$$\begin{aligned} &\limsup_{\lambda \rightarrow \infty} \int_{\mathbf{R}^n} \frac{\lambda}{\lambda^2 + |1 + \psi_1(z) + \psi_2(z)|^2} |\hat{\nu}(z)|^2 dz \\ &\leq \limsup_{\lambda \rightarrow \infty} \int_{\mathbf{R}^n} \frac{\lambda}{\lambda^2 + \gamma|1 + \psi_1(z)|^2} |\hat{\nu}(z)|^2 dz \\ &= \limsup_{\lambda \rightarrow \infty} \frac{1}{\sqrt{\gamma}} \int_{\mathbf{R}^n} \frac{\frac{1}{\sqrt{\gamma}}\lambda}{(\frac{1}{\sqrt{\gamma}}\lambda)^2 + |1 + \psi_1(z)|^2} |\hat{\nu}(z)|^2 dz \\ &= 0. \end{aligned}$$



Therefore,  $X_1 + X_2$  satisfies (H) by [9, Proposition 2.2].  $\square$

**Proof of Proposition 4.2.** It is easy to see that condition (i)  $\Rightarrow$  condition (ii)  $\Rightarrow$  condition (iii). In the following, we will prove that if condition (iii) is fulfilled, then any finite measure  $\nu$  of finite 1-energy w.r.t.  $X_1 + X_2$  has finite 1-energy w.r.t.  $X_1$ .

We denote by  $\psi$  the Lévy-Khintchine exponent of  $X_1 + X_2$ . Suppose that  $\nu$  is a finite measure of finite 1-energy w.r.t.  $X_1 + X_2$ , i.e.,

$$\int_{\mathbf{R}^n} \frac{1 + \operatorname{Re}\psi(z)}{|1 + \psi(z)|^2} |\hat{\nu}(z)|^2 dz = \int_{\mathbf{R}^n} \operatorname{Re} \left( \frac{1}{1 + \psi(z)} \right) |\hat{\nu}(z)|^2 dz < \infty. \quad (4.7)$$

By (4.1), for any  $z \in \mathbf{R}^n$ , we have

$$\begin{aligned} \operatorname{Re} \left( \frac{1}{1 + \psi(z)} \right) &= \frac{1}{1 + \operatorname{Re}\psi_1(z) + \operatorname{Re}\psi_2(z) + \frac{(\operatorname{Im}\psi_1(z) + \operatorname{Im}\psi_2(z))^2}{1 + \operatorname{Re}\psi_1(z) + \operatorname{Re}\psi_2(z)}} \\ &\geq \frac{1}{1 + \operatorname{Re}\psi_1(z) + \operatorname{Re}\psi_2(z) + \frac{2(\operatorname{Im}\psi_1(z))^2 + 2(\operatorname{Im}\psi_2(z))^2}{1 + \operatorname{Re}\psi_1(z) + \operatorname{Re}\psi_2(z)}} \\ &\geq \frac{1}{1 + \operatorname{Re}\psi_1(z) + \frac{2(\operatorname{Im}\psi_1(z))^2}{1 + \operatorname{Re}\psi_1(z)} + c \left( 1 + \operatorname{Re}\psi_1(z) + \frac{(\operatorname{Im}\psi_1(z))^2}{1 + \operatorname{Re}\psi_1(z)} \right) + \frac{2c(1 + \operatorname{Re}\psi_1(z) + \operatorname{Re}\psi_2(z)) \left( 1 + \operatorname{Re}\psi_1(z) + \frac{(\operatorname{Im}\psi_1(z))^2}{1 + \operatorname{Re}\psi_1(z)} \right)}{1 + \operatorname{Re}\psi_1(z) + \operatorname{Re}\psi_2(z)}} \\ &= \frac{1}{(1 + 3c) + (1 + 3c)\operatorname{Re}\psi_1(z) + (2 + 3c)\frac{(\operatorname{Im}\psi_1(z))^2}{1 + \operatorname{Re}\psi_1(z)}} \\ &\geq \frac{1}{2 + 3c} \cdot \frac{1}{1 + \operatorname{Re}\psi_1(z) + \frac{(\operatorname{Im}\psi_1(z))^2}{1 + \operatorname{Re}\psi_1(z)}} \\ &= \frac{1}{2 + 3c} \operatorname{Re} \left( \frac{1}{1 + \psi_1(z)} \right). \end{aligned} \quad (4.8)$$

By (4.7) and (4.8), we obtain that

$$\int_{\mathbf{R}^n} \operatorname{Re} \left( \frac{1}{1 + \psi_1(z)} \right) |\hat{\nu}(z)|^2 dz \leq (2 + 3c) \int_{\mathbf{R}^n} \operatorname{Re} \left( \frac{1}{1 + \psi(z)} \right) |\hat{\nu}(z)|^2 dz < \infty.$$

Therefore,  $\nu$  has finite 1-energy w.r.t.  $X_1$ .  $\square$

**Proof of Corollary 4.3.** We denote by  $U_{X_1}^1$  the 1-resolvent of  $X_1$ . By Assumption (i), for any finite measure  $\nu$ ,  $U_{X_1}^1 \nu$  is bounded. Hence  $U_{X_1}^1 \nu$  has finite 1-energy w.r.t.  $X_1$  by [13, Remark]. The corollary is therefore a direct consequence of Theorem 4.1.

**Proof of Proposition 4.5.** We define  $A(z) = 1 + \operatorname{Re}\psi(z)$  and  $B(z) = |1 + \psi(z)|$  for  $z \in \mathbf{R}^n$ . Then  $A(z) = 1 + \operatorname{Re}\psi_1(z) + \operatorname{Re}\psi_2(z)$  and  $B(z) = |1 + \psi_1(z) + \psi_2(z)|$ . We assume without loss of generality that  $f(1) = 1/3$ . Note that  $B(z) > 3\sqrt{2}A(z)f(A(z))$  implies that  $|\operatorname{Im}\psi(z)| > A(z)$  and

$|\operatorname{Im}\psi(z)| > B(z)/\sqrt{2}$ . Since  $|\operatorname{Im}\psi_2| \leq A(z)f(A(z))$ , we know that if  $|\operatorname{Im}\psi(z)| > 3A(z)f(A(z))$ , then  $|\operatorname{Im}\psi_1(z)| > 2A(z)f(A(z))$  and hence  $|\operatorname{Im}\psi_1(z)| \geq 2|\operatorname{Im}\psi_2(z)|$ . Thus

$$\begin{aligned} (\operatorname{Im}\psi(z))^2 &= (\operatorname{Im}\psi_1(z) + \operatorname{Im}\psi_2(z))^2 \\ &\geq (|\operatorname{Im}\psi_1(z)| - |\operatorname{Im}\psi_2(z)|)^2 \\ &\geq \frac{1}{4}(\operatorname{Im}\psi_1(z))^2. \end{aligned}$$

Note that  $|\operatorname{Im}\psi_1(z)| > 2A(z)f(A(z))$  implies that  $|\operatorname{Im}\psi_1(z)| > \frac{2}{3}(1 + \operatorname{Re}\psi_1(z))$  and  $|\phi_{12}(z)| \geq |\operatorname{Im}\psi_1(z)|/2$ . Then, by the fact that  $A(z) \leq c(1 + |z|^2)$  for some constant  $c > 0$  and the dominated convergence theorem, we obtain that

$$\begin{aligned} &\sum_{k=1}^{\infty} \int_{\{B(z) > 3\sqrt{2}A(z)f(A(z)), k \leq \frac{|\operatorname{Im}\psi(z)|}{A(z)} < k+1, A(z) \leq \lambda < (k+1)|\operatorname{Im}\psi(z)|\}} \frac{\lambda}{\lambda^2 + (\operatorname{Im}\psi(z))^2} |\hat{\nu}(z)|^2 dz \\ &\leq \sum_{k=1}^{\infty} \int_{\{|\operatorname{Im}\psi(z)| > 3A(z)f(A(z)), k \leq \frac{|\operatorname{Im}\psi(z)|}{A(z)} < k+1, A(z) \leq \lambda < (k+1)|\operatorname{Im}\psi(z)|\}} \frac{\lambda}{\lambda^2 + \frac{1}{4}(\operatorname{Im}\psi_1(z))^2} |\hat{\nu}(z)|^2 dz \\ &\leq \sum_{k=1}^{\infty} \int_{\{|\operatorname{Im}\psi(z)| > 3A(z)f(A(z)), k \leq \frac{|\operatorname{Im}\psi(z)|}{A(z)} < k+1, A(z) \leq \lambda < (k+1)|\operatorname{Im}\psi(z)|\}} \frac{2}{|\operatorname{Im}\psi_1(z)|} |\hat{\nu}(z)|^2 dz \\ &\leq \sum_{k=1}^{\infty} \int_{\{|\operatorname{Im}\psi_1(z)| > 2A(z)f(A(z)), k \leq \frac{|\operatorname{Im}\psi(z)|}{A(z)} < k+1, A(z) \leq \lambda < (k+1)|\operatorname{Im}\psi(z)|\}} \frac{4|\operatorname{Im}\psi_1(z)|}{2(\operatorname{Im}\psi_1(z))^2} |\hat{\nu}(z)|^2 dz \\ &\leq \sum_{k=1}^{\infty} \int_{\{k \leq \frac{|\operatorname{Im}\psi(z)|}{A(z)} < k+1, A(z) \leq \lambda < (k+1)|\operatorname{Im}\psi(z)|\}} \frac{8|\phi_{12}(z)|}{(\frac{2}{3}(1 + \operatorname{Re}\psi_1(z))^2 + (\operatorname{Im}\psi_1(z))^2)} |\hat{\nu}(z)|^2 dz \\ &\leq \sum_{k=1}^{\infty} \int_{\{k \leq \frac{|\operatorname{Im}\psi(z)|}{A(z)} < k+1, \lambda < (k+1)^2 A(z)\}} \frac{18|\phi_{12}(z)|}{(1 + \operatorname{Re}\psi_1(z))^2 + (\operatorname{Im}\psi_1(z))^2} |\hat{\nu}(z)|^2 dz \\ &\leq \sum_{k=1}^{\infty} \int_{\{k \leq \frac{|\operatorname{Im}\psi(z)|}{A(z)} < k+1, \lambda < c(k+1)^2(1+|z|^2)\}} \frac{18|\phi_{12}(z)|}{(1 + \operatorname{Re}\psi_1(z))^2 + (\operatorname{Im}\psi_1(z))^2} |\hat{\nu}(z)|^2 dz \\ &\rightarrow 0 \text{ as } \lambda \rightarrow \infty. \end{aligned}$$

Therefore,  $X_1 + X_2$  satisfies (H) by [10, Theorem 4.3].  $\square$

### 4.3 Examples

**Example 4.7** Let  $X_1$  and  $X_2$  be two independent Lévy processes on  $\mathbf{R}^n$ . We denote by  $\psi_1$  and  $\psi_2$  the Lévy-Khintchine exponents of  $X_1$  and  $X_2$ , respectively. Following Blumenthal and Gettoor [2], we define the indices:

$$\begin{aligned} \beta_1'' &:= \sup \left\{ \alpha \geq 0 : \frac{\operatorname{Re}\psi_1(z)}{|z|^\alpha} \rightarrow \infty \text{ as } |z| \rightarrow \infty \right\}, \\ \beta_2 &:= \inf \left\{ \alpha > 0 : \int_{\{|x| < 1\}} |x|^\alpha \nu_2(dx) < \infty \right\}, \end{aligned}$$

where  $\nu_2$  is the Lévy measure of  $X_2$ . We will prove below that if  $X_1$  satisfies (H) and  $\beta_2 < \beta_1''$ , then  $X_1 + X_2$  satisfies (H).

We fix a  $\beta \in (\beta_2, \beta_1'')$ . Then

$$\lim_{|z| \rightarrow \infty} \frac{\operatorname{Re} \psi_1(z)}{|z|^\beta} = \infty. \quad (4.9)$$

By [2, Theorem 3.2], we get

$$\lim_{|z| \rightarrow \infty} \frac{|\psi_2(z)|}{|z|^\beta} = 0. \quad (4.10)$$

(4.9) and (4.10) imply that there exists a constant  $c > 0$  such that

$$|\psi_2(z)| \leq c(1 + \operatorname{Re} \psi_1(z)), \quad \forall z \in \mathbf{R}^n.$$

By the assumption that  $\beta_2 < \beta_1''$ , we get  $\beta_1'' > 0$ . By (4.9) and [7], we know that  $X_1$  and hence  $X_1 + X_2$  have transition densities. Therefore,  $X_1 + X_2$  satisfies (H) by Theorem 4.1 and Proposition 4.2.

**Example 4.8** Suppose that  $\mu$  is a Lévy measure on  $(0, \infty)$  satisfying  $\int_{(0,1)} x \mu(dx) = +\infty$ ,  $\nu$  is a symmetric Lévy measure on  $\mathbf{R} \setminus \{0\}$ , and  $a \in \mathbf{R}$ . Let  $X$  be a Lévy process on  $\mathbf{R}$  with the Lévy-Khintchine exponent  $(a, 0, \mu + \nu)$ .

(i) If  $\int_{\{|x| < 1\}} |x| \nu(dx) < \infty$ , then  $X$  satisfies (H) by Kesten [12, Theorem 1(f)].

(ii) If  $\int_{\{|x| < 1\}} |x| \nu(dx) = \infty$  and the restriction of  $\mu$  on  $(0, \delta)$  is absolutely continuous w.r.t. the Lebesgue measure for some constant  $\delta$  ( $0 < \delta < 1$ ), then  $X$  satisfies (H). In fact, let  $X_1$  be a Lévy process on  $\mathbf{R}$  with the Lévy-Khintchine exponent  $(a, 0, \mu)$ . Then,  $X_1$  has transition densities (cf. [14, Theorem 27.7]) and bounded resolvent densities (see Remark 4.4(ii)). It follows that  $X$  has transition densities. Therefore,  $X$  satisfies (H) by Corollary 4.3.

Before presenting the next example, we recall the definition of type- $(\alpha, \beta)$  subordinator which is introduced in [9].

**Definition 4.9** ([9, Definition 4.1]) Let  $0 < \alpha < \beta < 1$ . A pure jump subordinator  $X$  is said to be of type- $(\alpha, \beta)$  if the Lévy measure of  $X$  has density, which is denoted by  $\rho$ , and there exists a constant  $c > 1$  such that

$$\frac{1}{cx^{1+\alpha}} \leq \rho(x) \leq \frac{c}{x^{1+\beta}}, \quad \forall x \in (0, 1].$$

Up to now it is still unknown if any pure jump subordinator of type- $(\alpha, \beta)$  satisfies (H). In [9], we have shown that any pure jump subordinator of type- $(\alpha, \beta)$  can be decomposed into the summation of two independent pure jump subordinators of type- $(\alpha, \beta)$  such that both of them satisfy (H) (see [9, Theorem 4.2]).

**Example 4.10** Let  $0 < \alpha_1 < \beta_1 < \alpha < \beta < 1$ . Suppose that  $X_1$  is a pure jump subordinator of type- $(\alpha, \beta)$  satisfying (H) and  $X_2$  is a pure jump subordinator of type- $(\alpha_1, \beta_1)$  which is independent of  $X_1$ . We will prove below that both  $X_1 + X_2$  and  $X_1 - X_2$  satisfy (H).

We denote by  $\psi_1$  and  $\psi_2$  the Lévy-Khintchine exponents of  $X_1$  and  $X_2$ , respectively. Note that  $\overline{\psi_2}$  is the Lévy-Khintchine exponent  $-X_2$ . By [9, (4.5) and (4.6)], we find that there exist two positive constants  $c_1$  and  $c_2$  such that

$$1 + \operatorname{Re}\psi_1(z) \geq 1 + c_1|z|^\alpha, \text{ for all } |z| \geq 1, \quad (4.11)$$

and

$$|\psi_2(z)| \leq c_2|z|^{\beta_1}, \text{ for all } |z| \geq 1.$$

Hence there exists a constant  $c > 0$  such that

$$|\psi_2(z)| \leq c(1 + \operatorname{Re}\psi_1(z)), \quad \forall z \in \mathbf{R}.$$

By (4.11) and [7], we know that  $X_1$  has transition densities and thus both  $X_1 + X_2$  and  $X_1 - X_2$  have transition densities. Therefore, both  $X_1 + X_2$  and  $X_1 - X_2$  satisfy (H) by Theorem 4.1 and Proposition 4.2.

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